Three-Phase, Pulse-Width Modulation, Controlled Inverter Circuits

The skeleton three-phase inverter circuits of Fig. 10.2 apply not only for step-wave operation but also for the switching technique known as pulse width modulation (PWM). Section 8.4 gives a brief account of the principles of PWM—this is subsumed within the present chapter.

11.1 SINUSOIDAL PULSE WIDTH MODULATION

A periodic (carrier) waveform of any wave shape can be modulated by another periodic (modulating) waveform of any other wave shape, of lower frequency. For most wave-shape combinations of a carrier waveform modulated by a modulating waveform, however, the resultant modulated waveform would not be suitable for either power applications or for information transmission.

For induction motor speed control the motor voltage waveforms should be as nearly sinusoidal as possible. If nonsinusoidal voltages are used, as with most inverter drives, it is preferable to use waveforms that do not contain low-order harmonics such as the fifth and the seventh because these can cause torque ripple disturbances, especially at low speeds. The lower order harmonics of a modulated voltage wave can be greatly reduced if a sinusoidal modulating signal modulates a triangular carrier wave. The pulse widths then cease to be uniform as in Fig. 8.5 but become sinusoidal functions of the angular pulse position, as in Fig. 11.1.

With sinusoidal PWM the large look-up tables of precalculated values needed...
for harmonic elimination in step-wave inverters (Sec. 8.4.3) are avoided. Inverter control becomes complex with sinusoidal PWM, and the switching losses are much greater than for conventional six-step operation of step-wave inverters. For SCR and GTO devices the switching frequencies are usually in the range 500–2500 Hz, and for power transistors the frequency can be as high as 10 kHz.

### 11.1.1 Sinusoidal Modulation with Natural Sampling

The principle of sinusoidal PWM was discussed in Sec. 8.4 and is followed up in this present section, with reference to Fig. 11.2. A sinusoidal modulating signal \( v_m (\omega t) = V_m \sin \omega t \) is applied to a single-sided triangular carrier signal \( v_c (\omega_c t) \), of maximum height \( V_c \), to produce the output modulated wave \( v_o (\omega t) \) of the same frequency as the modulating wave. For the example shown, the modulation ratio \( M \), defined as \( V_m/V_c \), is 0.75 and the frequency ratio \( p \), defined as \( f_c/f_m \) or \( f_c/f \), has the value 12 (Fig. 11.2). The peak value \( \pm V \) of the output wave is determined by the dc source voltage.

### 11.1.2 Three-Phase Sinusoidal PWM with Natural Sampling

The basic skeleton inverter circuit, with a wye-connected load, is shown in Fig. 11.3. This has exactly the same general form as the inverter circuits of Chapter 10, except that a midpoint of the dc supply is shown explicitly. For three-phase operation the triangular carrier wave is usually doubled sided, being symmetrical and without dc offset. Each half wave of the carrier is then an identical isosceles

---

**Fig. 1** Principle of sinusoidal modulation of triangular carrier wave [20].
triangle. Waveforms for a three-phase system are shown in Fig. 11.4 in which frequency ratio \( p = 9 \) and modulation ratio \( M \) is almost unity. For balanced three-phase operation, \( p \) should be an odd multiple of 3. The carrier frequency is then a triplen of the modulating frequency so that the output-modulated waveform does not contain the carrier frequency or its harmonics. In Fig. 11.4 the three-phase, modulated output voltages are seen to be identical but with a mutual phase displacement of 120° or one-third of the modulating voltage cycle, and the peak level is determined by the dc source level.

The order of harmonic components \( k \) of the modulated waveform are given by

\[
k = np \pm m
\]  
(11.1)

where \( n \) is the carrier harmonic order and \( m \) is the carrier side band. The major harmonic orders are shown in Table 11.1 for several values of \( p \). At \( p = 15 \), for example, the lowest significant harmonic is \( k = p - 2 = 13 \), and this is of much higher order than the harmonics \( k = 5, 7 \), obtained with a six-step waveform. It is found that the \( 2p \pm 1 \) harmonics are dominant in magnitude for values of
modulation ratio up to about $M = 0.9$. When $p > 9$, the harmonic magnitudes at a given value of $M$ are independent of $p$.

Fourier analysis of a sinusoidally pulse width modulated waveform is very complex. The $n$th harmonic phase voltage component for a waveform such as those of Fig. 11.4 is given by an expression of the form

$$v_n = \frac{MV_{dc}}{2} \cos \omega_m t$$

\[+ \frac{2V_{dc}}{\pi} \sum_{m=1}^{\infty} J_0 \left( \frac{mM \pi}{2} \right) \sin \left( \frac{m\pi}{2} \right) \cos(m\omega_c t) \]

\[+ \frac{2V_{dc}}{\pi} \sum_{m=1}^{\infty} \sum_{n=\pm 1}^{\infty} \frac{J_n \left( \frac{mM \pi}{2} \right)}{m} \sin \left( \frac{(m+n)\pi}{2} \right) \cos(m\omega_c t + n\omega_c t) \] (11.2)

In Eq. (11.2) the terms $J_0$, and $J_n$ represent first-order Bessel functions, $\omega_m$ is the fundamental frequency of the modulating and output waves, and $\omega_c$ is the carrier frequency.

The first term of Eq. (11.2) gives the amplitude of the fundamental frequency output component, which is proportional to modulation index $M$, in the range $0 \leq M \leq 1$, for all values of $p \geq 9$. 

---

**Fig. 3** Basic skeleton inverted circuit [20].
FIG. 4 Voltage waveforms for a three-phase sinusoidal PWM inverter, $m = 1 \ p = 9$: (a) timing waveforms, (b)–(d) pole voltages, and (e) output line voltage [21].

<table>
<thead>
<tr>
<th>Frequency ratio</th>
<th>$n = 1$</th>
<th>$n = 2$</th>
<th>$n = 3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m = 2$</td>
<td>$m = 1$</td>
<td>$m = 2$</td>
<td></td>
</tr>
<tr>
<td>$p$</td>
<td>$p \pm 2$</td>
<td>$2p \pm 1$</td>
<td>$3p \pm 2$</td>
</tr>
<tr>
<td>3</td>
<td>5, 1</td>
<td>7, 5</td>
<td>11, 7</td>
</tr>
<tr>
<td>9</td>
<td>11, 7</td>
<td>19, 17</td>
<td>29, 25</td>
</tr>
<tr>
<td>15</td>
<td>17, 13</td>
<td>31, 29</td>
<td>47, 43</td>
</tr>
<tr>
<td>21</td>
<td>23, 19</td>
<td>43, 41</td>
<td>65, 61</td>
</tr>
</tbody>
</table>

TABLE 11.1 Major Harmonics in a PWM Waveform with Naturally Sampled Sinusoidal Modulation [20]
\[ V_{L_1(\text{peak})} = \frac{MV_{dc}}{2} \]  \hspace{1cm} (11.3)

The corresponding rms value of the fundamental component of the modulated line-to-line voltage \( V_{L_1} \) (rms) is given by

\[ V_{L_1(\text{rms})} = \frac{\sqrt{3}V_{L_1(\text{peak})}}{\sqrt{2}} = \frac{\sqrt{3}}{2\sqrt{2}} MV_{dc} = 0.612MV_{dc} \]  \hspace{1cm} (11.4)

There is no straightforward analytical expression for the related values of the higher harmonic voltage components. Calculated values of these, in the range \( 0 \leq M \leq 1 \) with \( p > 9 \), are given in Fig. 11.5 [21]. If the modulating wave peak amplitude is varied linearly with the modulating frequency \( f_m \), then ratio \( Mf_m \) is constant. A waveform having a constant ratio of fundamental voltage to frequency is particularly useful for ac motor speed control.

**Fig. 5** Harmonic component voltages (relative to peak fundamental value) for sinusoidal PWM with natural sampling \( p > 9 \) [21].

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The second term of Eq. (11.2) describe harmonics at the carrier frequency and its multiples, while the third term refer to sidebands around each multiple of the carrier frequency.

### 11.1.3 Overmodulation in Sinusoidal PWM Inverters

Increase of the fundamental component of the modulated output voltage $V_1$, beyond the $M = 1$ value, is possible by making $M > 1$, but $V_1$ is then no longer proportional to $M$ (Fig. 11.6). In this condition of overmodulation the process of natural sampling no longer occurs. Some intersections between the carrier wave and the modulating wave are lost, as illustrated in Fig. 11.7. The result is that some of the pulses of the original PWM wave are dropped in the manner shown in Fig. 11.8.

In the extreme, when $M$ reaches the value $M = 3.24$, the original forms of PWM waveform in Fig. 11.4 are lost. The phase voltages then revert to the quasi-square wave shape of Table 10.1, Fig. 10.5, and Fig. 10.7 in which harmonics of orders 5 and 7 reappear. Variation of the fundamental output voltage versus modulation ratio $M$ is shown in Fig. 11.6. For pulse voltage $V_{dc}$ (i.e., twice the value given in Fig. 11.3) the rms fundamental line value of the quasi-square wave is

$$V_{1\text{rms}} = \frac{4 \sqrt{3}}{2 \sqrt{2}} V_{dc} = \frac{\sqrt{6}}{\pi} V_{dc} = 0.78 V_{dc}$$

(11.5)

---

**FIG. 6** RMS fundamental line voltage (relative to $V_{dc}$) versus modulation ratio $M$ for sinusoidal modulation [20].
FIG. 7 Overmodulation of triangular carrier wave by a sinusoidal modulating wave, $M = 1.55$ [21].

FIG. 8 Example of pulse dropping due to overmodulation: (a) containing a minimum pulse and (b) minimum pulse dripped [20].
Overmodulation increases the waveform harmonic content and can also result in undesirable large jumps of $V_1$, especially in inverter switches with large dwell times.

Other options for increase of the fundamental output voltage beyond the $M = 1$ value, without increase of other harmonics, are to use a nonsinusoidal reference (modulating) wave such as a trapezoid or a sine wave plus some third harmonic component.

### 11.1.4 Three-Phase Sinusoidal PWM with Regular Sampling

As an alternative to natural sampling, the sinusoidal reference wave can be sampled at regular intervals of time. If the sampling occurs at instants corresponding to the positive peaks or the positive and negative peaks of the triangular carrier wave, the process is known as uniform or regular sampling. In Fig. 11.9a sample value of the reference sine wave is held constant until the next sampling instant when a step transition occurs, the stepped version of the reference wave becomes, in effect, the modulating wave. The resulting output modulated wave is defined by the intersections between the carrier wave and the stepped modulating wave.

When sampling occurs at carrier frequency, coincident with the positive peaks of the carrier wave (Fig. 11.9a), the intersections of adjacent sides of the carrier with the step wave are equidistant about the non sampled (negative) peaks. For all values of $M$ the modulated wave pulse widths are then symmetrical about the lower (nonsampled) carrier peaks, and the process is called symmetrical regular sampling. The pulse centers occur at uniformly spaced sampling times.

When sampling coincides with both the positive and negative peaks of the carrier wave (Fig. 11.9b), the process is known as asymmetrical regular sampling. Adjacent sides of the triangular carrier wave then intersect the stepped modulation wave at different step levels and the resultant modulated wave has pulses that are asymmetrical about the sampling point.

For both symmetrical and asymmetrical regular sampling the output modulated waveforms can be described by analytic expressions. The number of sine wave values needed to define a sampling step wave is equal to the frequency ratio $p$ (symmetrical sampling) or twice the frequency ratio, $2p$ (asymmetrical sampling). In both cases the number of sample values is much smaller than in natural sampling, which requires scanning at sampling instants every degree or half degree of the modulating sine wave.

The use of regular sampled PWM in preference to naturally sampled PWM requires much less ROM-based computer memory. Also, the analytic nature of regular sampled PWM waveforms makes this approach feasible for implementation using microprocessor-based techniques because the pulse widths are easy to calculate.
Some details of various forms of pulse width modulation, using a symmetrical triangular carrier wave, are given in Table 11.2.

11.2 PWM VOLTAGE WAVEFORMS APPLIED TO A BALANCED, THREE-PHASE, RESISTIVE-INDUCTIVE LOAD

A double-sided triangular carrier wave modulated by a sinusoid results in the pulse waveforms \( v_A \) and \( v_B \) of Fig. 11.10. If modulating signal \( v_{mB} \) is delayed
## Table 11.2  Techniques of Pulse Width Modulation (PWM) Using a Symmetrical Triangular Carrier Wave [20]

<table>
<thead>
<tr>
<th>Square-wave modulating (reference) signal</th>
<th>Sinusoidal modulating (reference) signal</th>
<th>Optimal (time variant) modulating signal</th>
<th>Suboptimal modulating signal</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Natural sampling</strong> (essentially analogue in nature – pulse widths not describable in analytic form)</td>
<td><strong>Regular (uniform) sampling</strong> (essentially digital in nature – pulse-widths describable in analytic form)</td>
<td>Total harmonic minimisation</td>
<td>Individual harmonic suppression</td>
</tr>
<tr>
<td><strong>Synchronous</strong></td>
<td><strong>Asynchronous</strong></td>
<td><strong>Symmetric</strong> (sample only at positive peaks of the carrier)</td>
<td><strong>Asymmetric</strong> (sample every half-cycle of the carrier)</td>
</tr>
<tr>
<td>Synchronous carrier frequency is an integer triplen multiple of reference frequency</td>
<td>Asynchronous carrier frequency is not an integer multiple of the reference frequency</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
FIG. 10  PWM voltage waveforms with sinusoidal natural sampling instants, $M = 0.75$, $p = 12$ [20].

120° with respect to $v_{mA}$ the resulting modulated wave $v_B$ is identical in form to $v_A$ but is also delayed by 120°. The corresponding line voltage $v_{AB} (= v_A - v_B)$ has a fundamental component that leads the fundamental component of $v_A$ by 30°, as in a sinusoidal balanced set of voltages. Note that the positive pulse pattern of $v_{AB}$ ($of$) is not quite the same as the negative pulse pattern, although the two areas are the same and give zero time average value. This issue is the subject of Example 11.1.

The application of a PWM voltage waveform to an inductive load results in a current that responds (very nearly) only to the fundamental component. The harmonics of a PWM waveform, including the fundamental, are a complicated function of the carrier frequency $\omega_c$, the modulating (output) frequency $\omega_m$, the carrier amplitude $V_c$, and the modulating wave amplitude $V_m$ (combined in the modulation index $M$), as indicated in Eq. (11.2).

Harmonic components of the carrier frequency are in phase in all three load phases and therefore have a zero sequence nature. With a star-connected load there are no carrier frequency components in the line voltages.

An approximate method of calculating the harmonic content of a PWM waveform is to use graphical estimation of the switching angles, as demonstrated
in Example 11.1. A precise value of the intersection angles between the triangular carrier wave and the sinusoidal modulating wave can be obtained by equating the appropriate mathematical expressions. In Fig. 11.10, for example, the modulating wave is synchronized to the peak value of the carrier wave. The first intersection \( p_1 \) between the carrier \( v_c (\omega_m t) \) and modulating wave \( v_{mA} (\omega_m t) \) occurs when

\[
\frac{V_m}{V_c} \sin \omega_m t = 1 - \frac{24}{\pi} \omega_m t
\]

Intersection \( p_2 \) occurs when

\[
\frac{V_m}{V_c} \sin \omega_m t = -3 + \frac{24}{\pi} \omega_m t
\]

This oscillating series has the general solution, for the \( N \)th intersection,

\[
p_N = (2N - 1)(-1)^{N+1} + (-1)^N \frac{2p}{\pi} \omega_m t
\]

where \( N = 1, 2, 3, \ldots, 24 \).

Expressions similar to Eqs. (11.6) and (11.7) can be obtained for all of the intersections, as shown in Example 11.1. Equations of the form Eqs. (11.6)–(11.8) are transcendental and require being solved by iteration.

### 11.3 PWM VOLTAGE WAVEFORMS APPLIED TO A three-phase induction motor

The basic differences of structure between the voltage source, step-wave inverter, such as that of Fig. 10.1, and the voltage source, PWM inverter are given in Fig. 11.11. A step-wave inverter uses a controlled rectifier to give a direct-voltage source of adjustable level at the input to the dc link. The voltage level of the inverter output is controlled by the adjustable \( V_{dc} \) link voltage, whereas the frequency is controlled independently by the gating of the inverter switches.

A PWM inverter uses a diode bridge rectifier to give a fixed level of \( V_{dc} \) at the dc link. Both the voltage and frequency of the inverter are simultaneously controlled by gating of the inverter switches (Fig. 11.11b). The complete assembly of rectifier stage, dc link, and inverter stage is shown in Fig. 11.12. Since the output voltage of the diode bridge rectifier is not a pure direct voltage, a filter inductor is included to absorb the ripple component.

The use of a fixed dc rail voltage means that several independent inverters can operate simultaneously from the same dc supply. At low power levels the use of transistor (rather than thyristor) switches permits fast switching action and fast current and torque transient response, compared with step-wave inverters.
FIG. 11  Basic forms of voltage source inverter (VSI): (a) step wave (or quasi-square wave) and (b) PWM [20].

FIG. 12  Main circuit features of PWM VSI with motor load [20].
Because the harmonic currents are small and can be made of relatively high order, compared with single-pulse or multiple-pulse modulation, and because the fundamental component is easily controlled, PWM methods are becoming increasingly popular for ac motor control. Although the harmonic currents may be small, however, the harmonic heating losses may be considerable through increase of the motor resistances due to the skin effect. PWM switching techniques are better suited to power transistor inverters than to thyristor inverters because the commutation losses due to the many switchings are then less significant. Above about 100 Hz the commutation losses with PWM switching become unacceptably large, and stepped-wave techniques are used in ac motor drives.

When PWM voltage waveforms are applied to an induction motor, the motor torque responds largely to the fundamental frequency component. Motor current harmonics are usually small and of high harmonic order, depending on the frequency ratio $p$.

The harmonics of the PWM applied voltage are often more significant than those of the consequent motor current. This has the result that the eddy-current and hysteresis iron losses, which vary directly with flux and with frequency, are often greater than the copper losses in the windings. The total losses due to harmonics in a PWM driven motor may exceed those of the comparable step-wave driven motor. It is a common practice that a PWM driven motor is derated by an amount of 5–10%.

Torque pulsations in a PWM drive are small in magnitude and are related to high harmonic frequencies so that they can usually be ignored. The input current waveform to a dc link-inverter drive is determined mostly by the rectifier action rather than by the motor operation. This has a wave shape similar to that of a full-wave, three-phase bridge with passive series resistance–inductance load, so that the drive operates, at all speeds, at a displacement factor near to the ideal value of unity.

### 11.4 WORKED EXAMPLES

**Example 11.1**  A double-sided triangular carrier wave of height $V_c$ is natural sampling modulated by a sinusoidal modulating signal $v_m(\omega t) = V_m \sin \omega t$, where $V_m = 0.6V_c$. The carrier frequency $\omega_c$ is 12 times the modulating frequency $\omega_m$. Sketch a waveform of the resultant modulated voltage and calculate its principal harmonic components and its rms value.

The waveforms are shown in Fig. 11.10. The phase voltage $v_A(\omega t)$ is symmetrical about $\pi/2$ radians and contains only odd harmonics. Since $v_A(\omega t)$ is antisymmetrical about $\omega t = 0$, the Fourier harmonics $a_n = 0$, so that the fundamental output component is in phase with the modulating voltage $v_m(\omega t)$.

It is necessary to determine the intersection points $p_1$ to $p_6$.

Point $p_1$: From Eq. (12.31), $p = 12$, $V_m/V_c = 0.6$ so that
0.6 \sin \omega t = 1 - \frac{24}{\pi} \omega t

which gives

\omega t = 7^\circ

Point p\textsubscript{2}:

\frac{V_m}{V_c} \sin \omega t = \frac{4N}{\pi} \omega t - 3

or

\omega t = 24^\circ

Point p\textsubscript{3}:

\frac{V_m}{V_c} \sin \omega t = 5 - \frac{4N}{\pi} \omega t

\omega t = 34.5^\circ

Point p\textsubscript{4}:

\frac{V_m}{V_c} \sin \omega t = \frac{4N}{\pi} \omega t - 7

\omega t = 56^\circ

Point p\textsubscript{5}:

\frac{V_m}{V_c} \sin \omega t = 9 - \frac{N}{\pi} \omega t

\omega t = 63^\circ

Point p\textsubscript{6}:

\frac{V_m}{V_c} \sin \omega t = \frac{4N}{\pi} \omega t - 11

\omega t = 87^\circ

For the first quarter cycle in Fig. 11.10 waveform \(v_A\) is given by

\[
v_A(\omega t) = \begin{cases} 24^\circ, 56^\circ, 87^\circ & 7^\circ, 34.5^\circ, 63^\circ, 90^\circ \\ 7^\circ, 34.5^\circ, 63^\circ & 0^\circ, 24^\circ, 56^\circ, 87^\circ \end{cases}
\]
Fourier coefficients $b_n$ are given by

$$b_n = \frac{4}{n\pi} \int_{0}^{\pi/2} v_A(\omega t) \sin n\omega t \, d\omega t$$

in this case,

$$b_n = \frac{4V}{n\pi} \left( -\cos n\omega \begin{bmatrix} 24^\circ, 56^\circ, 87^\circ \\ 7^\circ, 34.5^\circ, 63^\circ \\ 0^\circ, 24^\circ, 56^\circ, 87^\circ \end{bmatrix} + \cos n\omega \begin{bmatrix} 24^\circ, 56^\circ, 87^\circ \\ 0^\circ, 24^\circ, 56^\circ, 87^\circ \end{bmatrix} \right)$$

$$= \frac{8V}{n\pi} (\cos n7^\circ + \cos n34.5^\circ + \cos n63^\circ - \cos n24^\circ - \cos n56^\circ - \cos n87^\circ - 0.5)$$

It is found that the peak values of Fourier coefficient $b_n$ are

$$b_1 = 0.62 \text{ V}$$
$$b_3 = \frac{8V}{3\pi} (0.038) = 0.0323 \text{ V}$$
$$b_5 = \frac{8V}{5\pi} (0.102) = 0.052 \text{ V}$$
$$b_7 = \frac{8V}{7\pi} (0.323) = 0.118 \text{ V}$$
$$b_9 = \frac{8V}{9\pi} (0.876) = 0.248 \text{ V}$$
$$b_{11} = \frac{8V}{11\pi} (2.45) = 0.567 \text{ V}$$
$$b_{13} = \frac{8V}{13\pi} (2.99) = -0.585 \text{ V}$$
$$b_{15} = \frac{8V}{15\pi} (3.44) = 0.548 \text{ V}$$
$$b_{17} = \frac{8V}{17\pi} (1.52) = -0.23 \text{ V}$$
$$b_{19} = \frac{8V}{19\pi} (1.22) = -0.164 \text{ V}$$

The rms value of the waveform is

$$V_A = \frac{V}{\sqrt{2}} \sqrt{b_1^2 + b_3^2 + b_5^2 + \ldots + b_{19}^2}$$

$$= \frac{V}{\sqrt{2}} \sqrt{0.598} = 0.547 \text{ V}$$
The distortion factor is
\[
\text{Distortion factor} = \frac{b_1}{\sqrt{2}} = \frac{0.626}{\sqrt{2 \times 0.547}} = 0.809
\]

Note that the highest value harmonics satisfy an \( n = p \pm 1 \) relation and that the low-order harmonics have small values. This enhances the suitability of the waveform for ac motor speed control.

Example 11.2 The PWM voltage waveform \( v_A(\omega t) \) of Fig. 11.10 is generated by an inverter that uses a modulating frequency of 50 Hz. If the dc supply is 200 V, calculate the rms current that would flow if \( v_A(\omega t) \) was applied to a single-phase series \( R-L \) load in which \( R = 10 \Omega \) and \( L = 0.01 \) H.

At the various harmonic frequencies the load impedance is
\[
|Z_h| = \sqrt{R^2 + (n\omega L)^2}
\]
which gives
\[
\begin{align*}
Z_1 &= \sqrt{10^2 + (2\pi \times 50 \times 0.01)^2} = 10.48 \Omega \\
Z_3 &= 13.74 \Omega \\
Z_5 &= 18.62 \Omega \\
Z_7 &= 24.16 \Omega \\
Z_9 &= 30 \Omega \\
Z_{11} &= 35.97 \Omega
\end{align*}
\]

If the harmonic voltages of Example 11.1 are divided by the respective harmonic impedances above, one obtains the following peak current harmonics:
\[
\begin{align*}
I_1 &= \frac{V_1}{Z_1} = \frac{0.626 \times 200}{10.48} = 11.95 \text{ A} \\
I_3 &= 0.47 \text{ A} \\
I_5 &= 0.56 \text{ A} \\
I_7 &= 0.977 \text{ A} \\
I_9 &= 1.65 \text{ A} \\
I_{11} &= 3.15 \text{ A}
\end{align*}
\]
The harmonic sum is
\[
\sum I_h^2 = I_1^2 + I_3^2 + I_5^2 + \cdots + I_{19}^2 = 172 \text{ A}^2
\]
This has an r.m.s. value

\[ I = \sqrt{\frac{\sum I_n^2}{2}} = 9.27 \text{ A} \]

which compares with a fundamental rms current of \(11.95/\sqrt{2} = 8.45 \text{ A}\).

The current distortion factor is therefore \(8.45/9.27 = 0.912\), which is greater (i.e., better) than the corresponding voltage distortion factor of 0.809.

Example 11.3 The PWM voltage waveform \(v_A(\omega t)\) of Fig. 11.10 is applied to a single-phase series \(R-L\) circuit with \(R = 10 \Omega\) and \(L = 0.01 \text{ H}\). Voltage \(v_A(\omega t)\) has a frequency of 50 Hz and an amplitude \(V = 200 \text{ V}\). Deduce the waveform of the resulting current.

The waveform \(v_A(\omega t)\) is reproduced in Fig. 11.13. If a direct voltage \(V\) is applied across a series \(R-L\) circuit carrying a current \(I_o\), the subsequent rise of current satisfies the relation

\[ i(\omega t) = \frac{V}{R} \left(1 - e^{-t/\tau}\right) + I_o e^{-t/\tau} = V \frac{1}{R} \left(1 - e^{-t/\tau}\right) \]

where \(I_o\) is the value of the current at the switching instant. In this case \(\tau = L/\omega R = \pi/10\omega\) so that

\[ i(\omega t) = 20 - (20 - I_o)e^{-10/\pi\omega} \]

---

**Fig. 13** PWM waveforms with series \(R-L\) load, from Example 11.3 [20].
At switch-on \( I_o = 0 \) and the current starts from the origin. Consider the current values at the voltage switching points in Fig. 11.13. These time values are given in Example 11.1.

At point \( p_1 \),
\[
\omega t = 7^\circ = 0.122 \text{ radian} \\
i(\omega t) = -20(1 - 0.678) = -6.44 \text{ A}
\]

At point \( p_2 \),
\[
\omega t = 24^\circ - 7^\circ = 0.297 \text{ radian} \\
i(\omega t) = 20 - (20 + 6.44) \times 0.39 = 9.8 \text{ A}
\]

At point \( p_3 \),
\[
\omega t = 34.5^\circ - 24^\circ = 0.183 \text{ radian} \\
i(\omega t) = -20 - (-20 - 9.8) \times 0.56 = -3.3 \text{ A}
\]

At point \( p_4 \),
\[
\omega t = 56^\circ - 34.5^\circ = 0.375 \text{ radian} \\
i(\omega t) = 20 - (20 + 3.3) \times 0.303 = 12.94 \text{ A}
\]

At point \( p_5 \),
\[
\omega t = 63^\circ - 56^\circ = 0.122 \text{ radian} \\
i(\omega t) = -20 - (-20 - 12.94) \times 0.678 = 2.33 \text{ A}
\]

At point \( p_6 \),
\[
\omega t = 87^\circ - 63^\circ = 0.419 \text{ radian} \\
i(\omega t) = 20 - (20 - 2.33) \times 0.264 = 15.34 \text{ A}
\]

Similarly,

At point \( p_7 \), \( \omega t = 93^\circ - 87^\circ = 0.105 \text{ radian} \), \( i(\omega t) = 5.3 \text{ A} \).
At point \( p_8 \), \( \omega t = 116^\circ - 93^\circ = 0.401 \text{ radian} \), \( i(\omega t) = 15.9 \text{ A} \).
At point \( p_9 \), \( \omega t = 124.5^\circ - 116^\circ = 0.148 \text{ radian} \), \( i(\omega t) = 2.4 \text{ A} \).
At point \( p_{10} \), \( \omega t = 145.5^\circ - 124.5^\circ = 0.367 \text{ radian} \), \( i(\omega t) = 14.53 \text{ A} \).
At point \( p_{11} \), \( \omega t = 155^\circ - 145.5^\circ = 0.166 \text{ radian} \), \( i(\omega t) = 0.3 \text{ A} \).
At point \( p_{12} \), \( \omega t = 172.5^\circ - 155^\circ = 0.305 \text{ radian} \), \( i(\omega t) = 12.5 \text{ A} \).
At point \( p_{13} \), \( \omega t = 187.5^\circ - 172.5^\circ = 0.262 \text{ radian} \), \( i(\omega t) = -5.9 \text{ A} \).
At point \( p_{14} \), \( \omega t = 201^\circ - 187.5^\circ = 0.236 \text{ radian} \), \( i(\omega t) = 7.78 \text{ A} \).
At point \( p_{15} \), \( \omega t = 220.5^\circ - 201^\circ = 0.34 \text{ radian} \), \( i(\omega t) = -10.6 \text{ A} \).
At point $p_{16}$, $\omega t = 229.5^\circ - 220.5^\circ = 0.157$ radian, $i(\omega t) = 1.43$ A.

The time variation of the current, shown in Fig. 11.13, is typical of the current waveforms obtained with PWM voltages applied to inductive and ac motor loads.

Example 11.4  The PWM waveforms of Fig. 11.4 have a height of 240 V and are applied as the phase voltage waveforms of a three-phase, four-pole, 50-Hz, star-connected induction motor. The motor equivalent circuit parameters, referred to primary, are $R_1 = 0.32 \, \Omega$, $R = 0.18 \, \Omega$, $X_1 = X_2 = 1.65 \, \Omega$, and $X_m = $ large. Calculate the motor rms current at 1440 rpm. What are the values of the main harmonic currents?

A four-pole, 50-Hz motor has a synchronous speed

$$N_s = \frac{f \times 120}{p} = \frac{50 \times 120}{4} = 1500 \text{ rpm}$$

At a speed of 1440 rpm the per-unit slip is given by Eq. (9.3)

$$S = \frac{1500 \times 1440}{1500} = 0.04$$

It is seen from Fig. 11.4a that the PWM line voltage waveform of Fig. 11.4 is the value of the battery (or mean rectified) supply voltage $V_{dc}$ in the circuit of Fig. 11.3. The rms value of the per-phase applied voltage is therefore, from Eq. (11.3),

$$V_{\text{rms}} = 0.9 \times \frac{240}{2\sqrt{2}} = 76.4 \text{ V/phase}$$

The motor per-phase equivalent circuit is a series $R-L$ circuit. At 1440 rpm this has the input impedance

$$Z_{m_{1440}} = \left(0.32 + \frac{0.18}{0.04}\right) + j1.65$$

$$= 4.82 + j1.65 = 5.095 \angle 18.9^\circ \, \Omega/\text{phase}$$

Therefore,

$$\text{Therefore, } I_1 = \frac{76.4 \angle 0}{5.095 \angle 18.9} = 15 \angle -18.9^\circ \, \text{A/phase}$$

With $M = 0.9$ the dominant harmonics are likely to be those of order $2p \pm 1$ and $p \pm 2$. In this case, therefore, with $p = 9$, the harmonics to be considered are $n = 7$, $11$, $17$, and $19$. The harmonic slip values are

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For the \( n \)th harmonic currents the relevant input impedances to the respective equivalent circuit are

\[
Z_{i_{in}} = \left\{ \begin{array}{c}
0.32 + \frac{0.18}{0.863} + j7 \times 1.65 = 0.53 + j11.55 = 11.56 \Omega \\
0.32 + \frac{0.18}{1.087} + j11 \times 1.65 = 0.486 + j18.15 = 18.16 \Omega \\
0.32 + \frac{0.18}{1.056} + j17 \times 1.65 = 0.291 + j28.1 = 28.2 \ \Omega \\
0.32 + \frac{0.18}{0.949} + j19 \times 1.65 = 0.51 + j31.35 = 31.35 \ \Omega
\end{array} \right.
\]

The 7, 11, 17, and 19 order harmonic voltage levels have to be deduced from Fig. 11.5. For \( M = 0.9 \) the \( 2p \pm 1 \) and \( p \pm 2 \) levels have the value 0.26 of the peak fundamental value. The \( M = 1 \) value of the r.m.s. component of the fundamental phase voltage, from Eq. (11.3), is

\[
V_{rms} = \frac{240}{2\sqrt{2}} = 84.86 \text{ V/phase}
\]

Therefore, the rms harmonic voltage values

\[
V_7 = V_{11} = V_{17} = V_{19} = 0.26 \times 84.86 = 22 \text{ V/Phase}
\]

The appropriate rms harmonic phase currents are

\[
I_7 = \frac{22.1}{11.56} = 1.91\text{A} \quad I_{17} = \frac{22.1}{28.2} = 0.78\text{A} \\
I_{11} = \frac{22.1}{18.16} = 1.22\text{A} \quad I_{19} = \frac{22.1}{31.35} = 0.7\text{A}
\]

The total rms current is obtained by the customary square-law summation.
\[ I_{\text{rms}} = \sqrt{I_1^2 + I_2^2 + I_{11}^2 + I_{17}^2 + I_{19}^2 + \cdots} \]

\[ I_{\text{rms}}^2 = 15^2 + (1.91)^2 + (1.22)^2 + (0.78)^2 + (0.7)^2 \]

\[ = 225 + 3.65 + 1.49 + 0.6 + 0.49 \]

\[ = 231.23 \text{ A}^2 \]

\[ I_{\text{rms}} = \sqrt{231.23} = 15.25 \text{ A} \]

which is about 2\% greater than the fundamental value.

**PROBLEMS**

11.1 A single-sided triangular carrier wave of peak height \( V_c \) contains six pulses per half cycle and is modulated by a sine wave, \( v_m (\omega t) = V_m \sin \omega t \), synchronized to the origin of a triangular pulse. Estimate, graphically, the values of \( \omega t \) at which intersections occur between \( v_c (\omega t) \) and \( v_m (\omega t) \) when \( V_m = V_c \). Use these to calculate values of the harmonics of the modulated wave up to \( n = 21 \) and thereby calculate the rms value.

11.2 The modulated voltage waveform described in Problem 11.1 is applied to a series R-L load in which \( R = 25 \Omega \) and \( X_L = 50 \Omega \) at 50 Hz. If the constant height of the PWM voltage wave is 400 V, calculate the resulting current harmonics up to \( n = 21 \). Calculate the resultant rms current. Compare the value of the current distortion factor with the voltage distortion factor.

11.3 Calculate the power dissipation in the R-L series circuit of Problem 11.2. Hence calculate the operating power factor.

11.4 The PWM voltage waveform \( v_o (\omega t) \) if Fig. 11.2 is applied to the series R-L load, \( R = 25 \Omega \) and \( X_L = 50 \Omega \) at 50 Hz. If \( V = 250 \text{ V} \) and \( f = 50 \text{ Hz} \), reduce the waveform of the resulting current.

11.5 The PWM waveform \( v_o (\omega t) \) of Fig. 11.2 has a height \( V = 240 \text{ V} \) and is applied as the phase voltage waveform of a three-phase, four-pole, 50 Hz, star-connected induction motor. The motor equivalent circuit parameters, referred to primary turns, are \( R_1 = 0.32 \Omega \), \( R_2 = 0.18 \Omega \), \( X_1 = X_2 = 1.65 \Omega \), \( X_m = \text{large} \). Calculate the motor rms current at 1440 rpm. What are the values of the main harmonic currents?

11.6 For the induction motor of Problem 11.5 calculate the input power and hence the power factor for operation at 1440 rpm.
11.7 If only the fundamental current component results in useful torque production, calculate the efficiency of operation for the motor of Problem 11.5 at 1440 rpm. (a) neglecting core losses and friction and windage, (b) assuming that the core losses plus friction and windage are equal to the copper losses.